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THE STABILITY OF LINEAR PERIODIC HAMILTONIAN SYSTEMS UNDER NON-HAMILTONIAN PERTURBATIONS†

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A definition of strong stability and strong instability is proposed for a linear periodic Hamiltonian system of differential equations under a given non-Hamiltonian perturbation. Such a system is subject to the action of periodic perturbations: an arbitrary Hamiltonian perturbation and a given non-Hamiltonian one. Sufficient conditions for strong stability and strong instability are established. Using the linear periodic Lagrange equations of the second kind, the effect of gyroscopic forces and specified dissipative and non-conservative perturbing forces on strong stability and strong instability is investigated on the assumption that the critical relations of combined resonances are satisfied.

Because of the presence of dissipative forces, many systems are described by differential equations that may be considered Hamiltonian only to within a certain degree of accuracy. Dissipative and non-conservative forces are often introduced so as to stabilize the performance of various controlled systems. Moreover, a paradoxical effect has been observed and investigated ([1–3], and so on), consisting of the expansion of the unstable regions of combined resonances when the dissipative forces are increased. In that connection Kirpichnikov [4, 5] proposed an approach which, starting from already known effects due to the influence of dissipative forces on the stability of equilibrium positions in stationary Lagrangian systems [6, 7], extends the analysis to the case of almost-Hamiltonian linear periodic systems of differential equations.

A theorem proved in [5] provides sufficient conditions for the system considered below to be strongly stable or strongly unstable with respect to a given non-Hamiltonian perturbation. The theorem has been used [8, 9] to investigate the stability of the rotational motion of a composite satellite in a slightly elliptic orbit.

This paper presents a stronger version of that theorem. The following fact will be illustrated through examples. Suppose that the unperturbed periodic Hamiltonian system is strongly stable in Krein's sense [10–12], i.e. it is stable and remains stable under arbitrary periodic Hamiltonian perturbations, and suppose that when a given non-Hamiltonian periodic perturbation is applied the system becomes asymptotically stable. At combined resonances, the combined action of both factors may make the system unstable, i.e. it will not always be strongly stable in the sense defined in this paper. Conversely, if a non-Hamiltonian perturbation induces instability, the system may be stabilized by the addition of a suitable Hamiltonian term.

Thus, the paradoxical effect of [1–3] is increased further: the introduction of dissipative perturbing forces in a periodic Hamiltonian system, strongly stable in Krein's sense, may make the system unstable at combination resonances.

1. THE STRONG STABILITY AND STRONG INSTABILITY THEOREM

Consider a linear periodic Hamiltonian system of ordinary differential equations

$$\mathbf{x} = \mathbf{IHx}, \quad \mathbf{x} = (\mathbf{p}, \mathbf{q}) \in \mathbf{R}^{2n} \tag{1.1}$$

and a perturbed linear periodic system "close to" system (1.1):

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$$\mathbf{x}^{\cdot} = \mathbf{I}(\mathbf{H} + \varepsilon\mathbf{H}_1)\mathbf{x} + \varepsilon\boldsymbol{\Delta}\mathbf{x} + \mathbf{o}(\varepsilon)\mathbf{x}, \quad \mathbf{x} = (\mathbf{p}, \mathbf{q}) \in \mathbf{R}^{2n} \tag{1.2}$$

where $\varepsilon \geqslant 0$ is a small parameter, and $\mathbf{H}^T = \mathbf{H}$, $\mathbf{H}_1^T = \mathbf{H}_1$, $\boldsymbol{\Delta}$, $\mathbf{o}(\varepsilon)$ are real $2n \times 2n$ matrices which are piecewise continuous and T-periodic in the independent variable—the time t. The elements of $\mathbf{o}(\varepsilon)$ are of first order of smallness in ε. The matrices $\mathbf{I}$, $\mathbf{H}$ and $\boldsymbol{\Delta}$ may be written in partitioned form as

$$\mathbf{I} = \begin{vmatrix} \mathbf{O} & -\mathbf{E} \\ \mathbf{E} & \mathbf{O} \end{vmatrix}, \quad \mathbf{H} = \begin{vmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{vmatrix}, \quad \boldsymbol{\Delta} = \begin{vmatrix} \boldsymbol{\Delta}_{11} & \boldsymbol{\Delta}_{12} \\ \boldsymbol{\Delta}_{21} & \boldsymbol{\Delta}_{22} \end{vmatrix} \tag{1.3}$$

where $\{\mathbf{H}_{j\nu}, \boldsymbol{\Delta}_{j\nu}\}$ are $n \times n$ matrices, $\mathbf{H}_{11}$, $\mathbf{H}_{22}$ are symmetric and $\mathbf{E}$ is the $n \times n$ identity matrix. The phase space $\mathbf{R}^{2n} = \{\mathbf{x}\}$ of canonical variables $\mathbf{x} = (\mathbf{p}, \mathbf{q})$ of systems (1.1) and (1.2), and the spaces $\mathbf{R}^n = \{\mathbf{p}\}$ of generalized momenta $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{R}^n = \{\mathbf{q}\}$ of generalized coordinates $\mathbf{q} = (q_1, \ldots, q_n)$ are considered with the standard Euclidean structures; scalar products are denoted by ordinary parentheses. The Hamiltonian of the unperturbed system is $H = (\mathbf{Hx}, \mathbf{x})/2$. The matrix $\varepsilon\mathbf{H}_1$ denotes an arbitrary Hamiltonian perturbation and the matrix $\varepsilon\boldsymbol{\Delta}$ is a given non-Hamiltonian perturbation.

System (1.1) or (1.2) is said to be stable (unstable) if its trivial solution $\mathbf{x} \equiv \mathbf{O}$ is stable (unstable) in Lyapunov's sense.

Definition. System (1.1) is said to be strongly stable (strongly unstable) under the non-Hamiltonian perturbations $\varepsilon\boldsymbol{\Delta}\mathbf{x}$ if, for any real symmetric piecewise-continuous T-periodic matrix $\mathbf{H}_1$, system (1.2) is stable (unstable) for any sufficiently small $\varepsilon > 0$.

Our problem is to derive criteria for the strong stability and strong instability of system (1.1) under a given non-Hamiltonian perturbation.

Let $\boldsymbol{\Phi}(t, \varepsilon)$ be a fundamental matrix of system (1.2), normalized at zero, and let $\boldsymbol{\Phi}(t) = \boldsymbol{\Phi}(t, 0)$. Consider the operators Φ, Φ_ε: $\mathbf{R}^{2n} \to \mathbf{R}^{2n}$ corresponding to systems (1.1) and (1.2) over the period T. To the operator Φ: $\mathbf{R}^{2n} \to \mathbf{R}^{2n}$ there corresponds a symplectic matrix $\boldsymbol{\Phi} = \boldsymbol{\Phi}(T)$.

In the space $\mathbf{R}^{2n}$, we define a skew product

$$[\mathbf{x}, \mathbf{z}] = (\mathbf{Ix}, \mathbf{z}) \quad \forall \mathbf{x}, \mathbf{z} \in \mathbf{R}^{2n} \tag{1.4}$$

corresponding to the symplectic coordinates $\mathbf{p}$, $\mathbf{q}$, and a bilinear form

$$[\mathbf{x}, \mathbf{z}]_{\mathbf{J}} = (\mathbf{Jx}, \mathbf{z}) \quad \forall \mathbf{x}, \mathbf{z} \in \mathbf{R}^{2n} \tag{1.5}$$

$$\mathbf{J} = \int_0^T (\boldsymbol{\Phi}(t))^{\mathrm{T}} (\boldsymbol{\Delta}^T \mathbf{I} + \mathbf{I}\boldsymbol{\Delta}) \boldsymbol{\Phi}(t) dt, \quad \mathbf{J}^{\mathrm{T}} = -\mathbf{J} \tag{1.6}$$

If $\det \mathbf{J} \neq 0$, the bilinear form (1.5) will define another skew product in $\mathbf{R}^{2n}$.

Consider the standard complexification of $\mathbf{R}^{2n}$, obtained by changing to the $2n$-dimensional complex space $\mathbf{C}^{2n} = {}^{\mathbf{C}}\mathbf{R}^{2n}$. The scalar and skew products will be carried over from $\mathbf{R}^{2n}$ to $\mathbf{C}^{2n}$ not in the usual manner but by extending them by linearity over the field $\mathbf{C}^1$. The complexifications of the real linear operators defined in $\mathbf{R}^{2n}$ will be denoted by the same letters.

If system (1.1) is such that at least one of its multipliers

$$\mu_1, \ldots, \mu_k, \ \bar{\mu}_1, \ldots, \bar{\mu}_k, \quad k \leqslant n, \quad (\mu_j \neq \mu_\nu, \ \mu_j \neq \bar{\mu}_\nu \ \text{ for } j \neq \nu) \tag{1.7}$$

i.e. the eigenvalues of the operator Φ, differs from unity in absolute value, the question under consideration is trivial: the system is strongly unstable for any given non-Hamiltonian perturbation. From now on, therefore, we shall assume that the multipliers (1.7) all lie on the circle $|\mu| = 1$ in the complex plane. It is also important that all the results of this paper are obtained on the assumption that system (1.1) is stable (the elementary divisors of Φ are prime).

If μ is an r-fold multiplier of system (1.1), we let T^μ denote the corresponding r-dimensional complex invariant root subspace of Φ, putting $T_\mu = T^\mu$, if $\mu = \pm 1$ and $T_\mu = T^\mu + T^{\bar{\mu}}$ if $\mu \neq \pm 1$. Each invariant subspace T_μ of Φ has a basis $\{\mathbf{e}_j, \bar{\mathbf{e}}_j\}_1^r$ such that

$$[\mathbf{e}_j, \mathbf{e}_\nu] = 0; \quad [\mathbf{e}_j, \bar{\mathbf{e}}_\nu] = 0, \quad j \neq \nu; \quad [\mathbf{e}_j, \bar{\mathbf{e}}_j] \neq 0; \quad j, \nu = 1, 2, \ldots, r \tag{1.8}$$

and if the elementary divisors are prime, one can assume that $\Phi\mathbf{e}_j = \mu\mathbf{e}_j$, $j = 1, 2, \ldots, r$. Moreover (summing from $j = 1$ to $j = r$), we have $\operatorname{Re} T_\mu = \{\mathbf{x} = \Sigma(x_j\mathbf{e}_j + \bar{x}_j\bar{\mathbf{e}}_j) \mid x_j, \bar{x}_j \in \mathbf{C}^1\}$—the subspace of real vectors in T_μ, so that $\mathbf{x} = 2\Sigma(\xi_j\mathbf{r}_j - \eta_j\mathbf{s}_j)$, where $x_j = \xi_j + i\eta_j$, $\xi_j, \eta_j \in \mathbf{R}^1$, $\mathbf{e}_j = \mathbf{r}_j + i\mathbf{s}_j$, $\mathbf{r}_j, \mathbf{s}_j \in \mathbf{R}^{2n}$. The space $\mathbf{C}^{2n}$ may thus be expressed as a direct sum $\mathbf{C}^{2n} = T_{\mu 1} + \cdots + T_{\mu k}$ of even-dimensional Φ-invariant subspaces.

Later we will need some elements from the theory of Krein, Gel'fand and Lidskii. A multiplier μ of a symplectic mapping Φ: $\mathbf{R}^{2n} \to \mathbf{R}^{2n}$ is said to be sign-definite if the quadratic form $[\Phi\mathbf{x}, \mathbf{x}]$, $\mathbf{x} \in \operatorname{Re} T_\mu$ is sign-definite. The multipliers $+1, -1$ and multipliers with multiple elementary divisors cannot be sign-definite. For an r-fold multiplier μ with prime elementary divisors and such that $\gamma = \operatorname{Im}\mu \neq 0$ we have

$$[\Phi\mathbf{x},\mathbf{x}] = 2i\gamma\Sigma\ |x_j|^2[\mathbf{e}_j,\bar{\mathbf{e}}_j] = 4\gamma\Sigma\ (\xi_j^2+\eta_j^2)[\mathbf{r}_j,\mathbf{s}_j] \tag{1.9}$$

i.e. μ is sign-definite if and only if all the numbers $\{i[\mathbf{e}_j, \bar{\mathbf{e}}_j]\}_1^r$ have the same sign. A prime multiplier is always sign-definite. A symplectic transformation Φ of $\mathbf{R}^{2n}$ is said to be strongly stable if any sufficiently close symplectic transformation of the same space is also stable. A symplectic transformation Φ is strongly stable if and only if all its eigenvalues lie on the unit circle $|\mu| = 1$ and are all sign-definite.

A few examples will now illustrate the qualitative effects that may be caused by the combined action of Hamiltonian and non-Hamiltonian perturbations in cases of combined resonances. We will first carry out the analysis in the class of operators corresponding to systems (1.1) and (1.2) over one period, and then consider specific examples of the systems themselves. Putting $n = 2$, $T = 2\pi$, define

$$\Phi(2\pi,\varepsilon) = \mathbf{I}+\varepsilon\Phi_1,\quad \Phi_1 = \begin{vmatrix}\mathbf{O} & \sigma\mathbf{E}\\ \mathbf{O} & \mathbf{D}\end{vmatrix},\quad \mathbf{D} = \begin{vmatrix}0 & d_{12}\\ d_{21} & 0\end{vmatrix} \tag{1.10}$$

where σ, d_{12} and d_{21} are real constants. The matrix $\Phi(2\pi, \varepsilon)$ is symplectic for $\varepsilon > 0$ if and only if $\sigma = 0$, $d_{12} = d_{21}$. An easy analysis of the absolute values of the multipliers, i.e. the roots of the equation

$$\mu^4+(2-2\sigma\varepsilon-d_{12}d_{21}\varepsilon^2)\mu^2+(1-\sigma\varepsilon)^2 = 0 \tag{1.11}$$

shows that the operator Φ_ε is stable. In the unperturbed case, $\Phi(2\pi, 0) = \mathbf{I}$, we have two sign-definite double multipliers $\{-1, +1\}$, i.e. system (1.1) is stable in Krein's sense.

Example 1. A given non-Hamiltonian (index minus) perturbation: $d_{12} = d_{12}^- = 1$, $d_{21} = d_{21}^- = -1$, $\sigma = 0$ defining an unstable system (1.2) when $\varepsilon > 0$, $\mathbf{H}_1 \equiv \mathbf{O}$, may be stabilized by a suitable Hamiltonian parametric applied force. Thus, when this perturbation is combined with a Hamiltonian (index plus) perturbation $d_{12} = d_{12}^+ = 2$, $d_{21} = d_{21}^+ = 2$, $\sigma = 0$, the total perturbation will be $d_{12} = d_{12}^- + d_{12} = 3$, $d_{21} = d_{21}^- + d_{21}^+ = 1$, $\sigma = 0$, which makes system (1.2) stable when $\varepsilon > 0$. Thus, a system which is strongly stable in Krein's sense and unstable under a given non-Hamiltonian perturbation when $\varepsilon > 0$, $\mathbf{H}_1 \equiv \mathbf{O}$ may be stabilized by a Hamiltonian perturbation, i.e. it need not be strongly unstable in the sense of the definition adopted here.

Example 2. A given non-Hamiltonian perturbation $d_{12} = d_{12}^- = -1$, $d_{21} = d_{21}^- = -3$, $\sigma = 1/2$ defines an asymptotically stable system (1.2) when $\varepsilon > 0$, $\mathbf{H}_1 \equiv \mathbf{O}$. In combination with a Hamiltonian perturbation $d_{12}^+ = 2$, $d_{21}^+ = 2$, $\sigma = 0$ we get $d_{12} = d_{12}^- + d_{12}^+ = 1$, $d_{21} = d_{21}^- + d_{21}^+ = -1$, $\sigma = 1/2$ and system (1.2) becomes unstable when $\varepsilon > 0$. Thus, a system that is strongly stable in Krein's sense and asymptotically stable under a given non-Hamiltonian perturbation when $\varepsilon > 0$, $\mathbf{H}_1 \equiv \mathbf{O}$ need not be strongly stable in the sense of the definition adopted here.

Taking a specific example of systems (1.1) and (1.2), let us put $\mathbf{H}_1 \equiv \mathbf{O}$ and include all perturbations in the matrix $\mathbf{\Delta}$. System (1.1) will be

$$p_1^{\cdot} = -\omega_1 q_1,\quad p_2^{\cdot} = -\omega_2 q_2,\quad q_1^{\cdot} = \omega_1 p_1,\quad q_2^{\cdot} = \omega_2 p_2 \tag{1.12}$$

where the case corresponding to combined-difference resonance is

$$\omega_1 = \beta_1 = \tfrac{1}{4}+k_1,\quad \omega_2 = \beta_2 = \tfrac{1}{4}+k_2,\quad k_2 \geqslant k_1,\quad k_1 \in \mathbf{N} = \{0,1,2,\ldots\};\quad \beta_2-\beta_1 = k_2-k_1 \geqslant 0 \tag{1.13}$$

and that corresponding to combined-sum resonance is

$$\omega_1 = \beta_1 = \tfrac{1}{4}+k_1,\quad -\omega_2 = \beta_2 = \tfrac{3}{4}+k_2,\quad k_2 \geqslant k_1,\quad k_1 \in \mathbf{N};\quad \beta_2+\beta_1 = 1+k_1+k_2 \geqslant 1 \tag{1.14}$$

At simple resonance (1.13), $k_2 = k_1$, the effects under consideration may occur in linear systems with constant coefficients, i.e. $\mathbf{\Delta}$ = const. The matrix (1.10) corresponds to the perturbed system (1.2) with

$$\mathbf{\Delta}_{11} = -((4k_1+1)\pi\mathbf{D}+2\sigma\mathbf{E})/(8\pi), \quad \mathbf{\Delta}_{12} = (2\mathbf{D}+(4k_1+1)\sigma\pi\mathbf{E})/(8\pi) \tag{1.15}$$

$$\mathbf{\Delta}_{21} = (-2\mathbf{D}+(4k_1+1)\sigma\pi\mathbf{E})/(8\pi), \quad \mathbf{\Delta}_{22} = ((4k_1+1)\pi\mathbf{D}-2\sigma\mathbf{E})/(8\pi)$$

This system is equivalent to the following Lagrange equations of the second kind

$$q_1^{\cdot\cdot}/\omega_1 + \omega_1 q_1 = -\varepsilon\sigma q_1^{\cdot}/(2\omega_1\pi) + \varepsilon d_{12}q_2/(2\pi) + O(\varepsilon^2) \tag{1.16}$$

$$q_2^{\cdot\cdot}/\omega_1 + \omega_1 q_2 = -\varepsilon\sigma q_2^{\cdot}/(2\omega_1\pi) + \varepsilon d_{21}q_1/(2\pi) + O(\varepsilon^2)$$

In the general case of resonances (1.13) and (1.14), the matrix (1.10) corresponds to system (1.2), e.g. with a piecewise-smooth 2π-periodic matrix $\mathbf{\Delta}(t)$ defined over the period $t \in [0, 2\pi]$ by setting $c_i = \cos(\omega_i t)$, $s_i = \sin(\omega_i t)$, $i = 1, 2$, $\zeta = (1 - \cos t)/(2\pi)$

$$\mathbf{\Delta}_{11} = \zeta\begin{vmatrix} -s_1^2\sigma & -c_1s_2d_{12} \\ -c_2s_1d_{21} & -s_2^2\sigma \end{vmatrix}, \quad \mathbf{\Delta}_{12} = \zeta\begin{vmatrix} s_1c_1\sigma & c_1c_2d_{12} \\ c_1c_2d_{21} & s_2c_2\sigma \end{vmatrix}$$

$$\mathbf{\Delta}_{21} = \zeta\begin{vmatrix} c_1s_1\sigma & -s_1s_2d_{12} \\ -s_1s_2d_{21} & c_2s_2\sigma \end{vmatrix}, \quad \mathbf{\Delta}_{22} = \zeta\begin{vmatrix} -c_1^2\sigma & s_1c_2d_{12} \\ s_2c_1d_{21} & -c_2^2\sigma \end{vmatrix} \tag{1.17}$$

The equivalent Lagrange equations of the second kind for this system are rather cumbersome and will be omitted for brevity.

We now return to our investigation of the general problem.

Definition. We shall say that a multiplier μ of system (1.1) is of first class if both quadratic forms $[\mathbf{\Phi x}, \mathbf{x}]$, $[\mathbf{\Phi x}, \mathbf{x}]_\mathbf{J}$, $\mathbf{x} \in \operatorname{Re} T_\mu$, are sign-definite of different signs; a multiplier μ of system (1.1) will be of second class if the closure of the domain in which the quadratic form $[\mathbf{\Phi x}, \mathbf{x}]$ $\mathbf{x} \in \operatorname{Re} T_\mu$ is sign-definite (without the point $\mathbf{x} = \mathbf{O}$) is a subdomain of that in which the form $[\mathbf{\Phi x}, \mathbf{x}]_\mathbf{J}$, $\mathbf{x} \in \operatorname{Re} T_\mu$ is sign-definite with the same sign.

Not every multiplier is of first or second class. In particular, a necessary condition for μ to be of first class is that it should be sign-definite, i.e. that the corresponding quadratic form $[\mathbf{\Phi x}, \mathbf{x}]$, $\mathbf{x} \in \operatorname{Re} T_\mu$ should be sign-definite.

Theorem 1. If every multiplier $\mu_1, \ldots, \mu_k$ of the unperturbed system (1.1) is of first class, then system (1.1) is strongly stable under a given non-Hamiltonian perturbation. If at least one of them is of second class, the unperturbed system (1.1) is strongly unstable under a given non-Hamiltonian perturbation.

This is the fundamental theorem on the sufficient conditions for the strong stability and instability of system (1.1) under a given non-Hamiltonian perturbation. The proof makes use of discrete Lyapunov and Chetayev functions. If the multipliers $\mu_1(\varepsilon), \ldots, \mu_l(\varepsilon)$ of the perturbed system (1.2) tend continuously as $\varepsilon \to 0$ to a multiplier μ of system (1.1), one introduces the direct sum $T_\mu^\varepsilon = T_{\mu 1(\varepsilon)} + \ldots + T_{\mu l(\varepsilon)}$ of Φ_ε-invariant subspaces and, in the corresponding real space $\operatorname{Re} T_\mu^\varepsilon$, takes the quadratic form $[\mathbf{\Phi}_\varepsilon \mathbf{x}, \mathbf{x}]$, $\mathbf{x} \in \operatorname{Re} T_\mu$ as a Lyapunov or Chetayev function. The other changes needed in the proof given in [5] are obvious and will be omitted for brevity.

Remarks. 1. Strongly stable systems with a given non-Hamiltonian perturbation that satisfy the conditions of Theorem 1 are characterized by the property that the trivial solution is asymptotically stable for all sufficiently small $\varepsilon > 0$.

2. There is an important difference between this situation and that of [2], in which the terms "strong stability and instability of combined resonance frequencies" are considered for a different class of admissible perturbations: the arbitrary perturbations in that paper are of a higher order of smallness relative to the given ones; the latter, as in this paper, may be non-Hamiltonian. In addition, the unperturbed system there is gyroscopically disconnected.

3. Shnol' has shown that the sufficient conditions established in Theorem 1 for the strong stability of a system (1.1) under a given non-Hamiltonian perturbation are also necessary. His exposition is based on his own definition, which is equivalent to ours: system (1.1) is strongly stable under a given non-Hamiltonian perturbation $\varepsilon\mathbf{\Delta}$ if it satisfies the following conditions: (a) it is stable when $\varepsilon = 0$; (b) it is asymptotically stable when $\varepsilon > 0$, and (for sufficiently small ε) this asymptotic stability is preserved for any superimposed perturbation of order greater than ε; (c) both properties are preserved in some neighbourhood of the system, i.e. for $|\tilde{H} - H| < \delta$, where δ is independent of ε, and $\tilde{H}$ is the Hamiltonian of the new system.

4. If the sufficient conditions for strong stability (strong instability) under a given non-Hamiltonian perturbation are satisfied for system (1.1) with a Hamiltonian function H, they are also satisfied for any "nearby" T-periodic Hamiltonian systems, i.e. for $|\tilde{H} - H| < \delta$, where δ is independent of ε Thus these conditions guarantee stability (instability) in an extended sense—under finite Hamiltonian perturbations.

In the non-resonant case [4] the contribution to $|\mu_i(\varepsilon)| = \kappa_i\varepsilon + o(\varepsilon)$, $i = 1, 2, \ldots, n$, from the Hamiltonian part of the perturbation is of order $o(\varepsilon)$. Here the asymptotic stability (instability) of system (1.2) when $\mathbf{H}_1 \equiv \mathbf{O}$, as inferred by analysing the signs of $\kappa_1, \ldots, \kappa_n$, implies the strong stability (strong instability) of system (1.1) under a given non-Hamiltonian perturbation. Below, therefore, we shall consider the resonant case, assuming that at least one of the multipliers (1.7) is multiple and referring to the corresponding relations among the parameters of system (1.1) as the critical parametric resonance relations. Taking into account that the multipliers $+1$, -1 and multipliers with multiple elementary divisors can be of neither first nor second class, we exclude them from consideration. Now, for an r-fold multiplier μ, $\gamma = \operatorname{Im}\mu \neq 0$, we have

$$[\mathbf{\Phi x}, \mathbf{x}]_\mathbf{J} = 4\gamma \sum_{j,\nu=1}^{r} \{(\xi_j\xi_\nu + \eta_j\eta_\nu)[\mathbf{r}_\nu, \mathbf{s}_j]_\mathbf{J} + \xi_j\eta_\nu([\mathbf{s}_j, \mathbf{s}_\nu]_\mathbf{J} + [\mathbf{r}_j, \mathbf{r}_\nu]_\mathbf{J})\} \tag{1.18}$$

When $\mathbf{H} = \text{const}$, there is no need to find the matrix $\mathbf{\Phi}(t)$. Indeed, let μ be an r-fold multiplier of system (1.1). Then the set of eigenvalues of the matrix $\mathbf{IH}$ includes numbers $\pm i\beta_1, \ldots, \pm i\beta_r$, where $\beta_1 > 0, \ldots, \beta_r > 0$, so that $\mu = e^{i\beta_j T}$ or $e^{-i\beta_j T}$ for all $j = 1, 2, \ldots, r$. The corresponding eigenvectors of the matrices $\mathbf{IH}$ and $\mathbf{\Phi}$ are identical. It will be convenient to modify our notation, requiring henceforth that $\mathbf{e}_j$ be an eigenvector of $\mathbf{IH}$ belonging to the eigenvalue $i\beta_j$, $j = 1, 2, \ldots, r$. Then, for $j, \nu = 1, 2, \ldots, r$, we have $[\mathbf{e}_j, \bar{\mathbf{e}}_j] = i(\mathbf{He}_j, \bar{\mathbf{e}}_j)/\beta$ and

$$\begin{aligned} [\mathbf{e}_j, \mathbf{e}_\nu]_\mathbf{J} &= \int_0^T e^{i(\beta_j+\beta_\nu)t}((\mathbf{\Delta}^T\mathbf{I} + \dot{\mathbf{I}}\mathbf{\Delta})\mathbf{e}_j, \mathbf{e}_\nu)dt \\ [\mathbf{e}_j, \bar{\mathbf{e}}_\nu]_\mathbf{J} &= \int_0^T e^{i(\beta_j-\beta_\nu)t}((\mathbf{\Delta}^T\mathbf{I} + \mathbf{I\Delta})\mathbf{e}_j, \bar{\mathbf{e}}_\nu)dt \end{aligned} \tag{1.19}$$

2. APPLICATIONS OF THEOREM 1

First let $n = 2$. We also put $\mathbf{H} = \text{const}$, $\beta_1 > 0$, $\beta_2 > 0$ and, without loss of generality, $(\mathbf{He}_1, \bar{\mathbf{e}}_1) > 0$, $T = 2\pi$. We will refer to a matrix as positive-definite, positive-semidefinite or sign-variable if the corresponding quadratic form has the appropriate property. For the critical relations of combination resonances

$$|\beta_1 - \beta_2| = N, \quad N = 0, 1, 2, \ldots, \quad \chi = -2\sin(2\pi\beta_1) \neq 0 \tag{2.1}$$

$$\beta_1 + \beta_2 = N, \quad N = 1, 2, \ldots, \quad \chi = -2\sin(2\pi\beta_1) \neq 0 \tag{2.2}$$

we have, respectively

$$[\mathbf{\Phi x}, \mathbf{x}] = \chi[(\xi_1^2 + \eta_1^2)(\mathbf{He}_1, \bar{\mathbf{e}}_1)/\beta_1 + (\xi_2^2 + \eta_2^2)(\mathbf{He}_2, \bar{\mathbf{e}}_2)/\beta_2] \tag{2.3}$$

$$[\mathbf{\Phi x}, \mathbf{x}]_\mathbf{J} = \chi[(\xi_1^2 + \eta_1^2)a + (\xi_2^2 + \eta_2^2)b + 2d(\xi_1\xi_2 + \eta_1\eta_2) + 2c(\xi_2\eta_1 - \xi_1\eta_2)]$$

$$[\mathbf{\Phi x}, \mathbf{x}] = \chi[(\xi_1^2 + \eta_1^2)(\mathbf{He}_1, \bar{\mathbf{e}}_1)/\beta_1 - (\xi_2^2 + \eta_2^2)(\mathbf{He}_2, \bar{\mathbf{e}}_2)/\beta_2] \tag{2.4}$$

$$[\mathbf{\Phi x}, \mathbf{x}]_\mathbf{J} = \chi[(\xi_1^2 + \eta_1^2)a - (\xi_2^2 + \eta_2^2)b + 2e(\xi_1\xi_2 - \eta_1\eta_2) + 2f(\eta_1\xi_2 + \xi_1\eta_2)]$$

$$[\mathbf{e}_1, \bar{\mathbf{e}}_1]_\mathbf{J} = ai, \quad [\mathbf{e}_2, \bar{\mathbf{e}}_2]_\mathbf{J} = bi, \quad [\mathbf{e}_1, \bar{\mathbf{e}}_2]_\mathbf{J} = c + id, \quad [\mathbf{e}_1, \mathbf{e}_2]_\mathbf{J} = e + if \tag{2.5}$$

These formulae imply the following propositions.

Theorem 2. Assume that conditions (2.1) for combined-difference resonance are satisfied. System (1.1) is strongly stable under a given non-Hamiltonian perturbation if $a < 0$, $ab > c^2 + d^2$ and $\mathbf{H}$ is positive-definite; a sufficient condition for the system to be strongly unstable is that one of the following conditions holds: (1) $ab > c^2 + d^2$ and $\mathbf{H}$ is non-singular and sign-variable; (2) $a > 0, ab > c^2 + d^2$ and $\mathbf{H}$ is positive-semidefinite; (3) $c = 0, d = 0, a \geqslant 0, b \leqslant 0$ and $\mathbf{H}$ is sign-variable, $\beta_1 a(\mathbf{He}_2, \bar{\mathbf{e}}_2) \neq \beta_2 b(\mathbf{He}_1, \bar{\mathbf{e}}_1)$.

Theorem 3. Assume that conditions (2.2) for overall combination resonance are fulfilled. System (1.1) is strongly stable under a given non-Hamiltonian perturbation if $a < 0, ab < -e^2 - f^2$ and $\mathbf{H}$ is positive-definite; a sufficient condition for the system to be strongly unstable is that one of the following conditions holds: (1) $ab < -e^2 - f^2$ and $\mathbf{H}$ is positive-definite, (2) $a > 0, ab < -e^2 - f^2$ and $\mathbf{H}$ is sign-variable or singular, (3) $e = 0, f = 0, a \geqslant 0, b \geqslant 0$ and $\mathbf{H}$ is positive-definite, $\beta_1 a(\mathbf{He}_2, \bar{\mathbf{e}}_2) \neq \beta_2 b(\mathbf{He}_1, \bar{\mathbf{e}}_1)$.

Suppose that system (1.1) is strongly stable in Krein's sense, i.e. the quadratic form (1.9) is sign-definite. Then, if a given non-Hamiltonian perturbation is introduced, it may either remain strongly stable or become strongly unstable.

For example, for the difference resonance (2.1), if $ab > c^2 + d^2$, and in the case of the sum resonance (2.2) if $ab < -e^2 - f^2$, the system will remain strongly stable under a given non-Hamiltonian perturbation if $a < 0$ or, conversely, become strongly unstable if $a > 0$.

We will now analyse in greater detail the influence of forces of various kinds on strong stability and instability as defined here. Suppose that systems (1.1) or (1.2) are derived from Lagrange's equations of the second kind, so that the Hamiltonian function H is the Legendre transform of the Lagrange function

$$L_\varepsilon = L + \varepsilon L_1 + (\mathbf{o}(\varepsilon)\mathbf{y}, \mathbf{y}), \quad \mathbf{L} = \tfrac{1}{2}[(\mathbf{Bq}^\cdot, \mathbf{q}^\cdot) + (\mathbf{Cq}, \mathbf{q}^\cdot) - (\mathbf{Pq}, \mathbf{q})] \tag{2.6}$$

where $L_1 = L_1(t, \mathbf{q}, \mathbf{q}^\cdot)$ is a quadratic form in the variable $\mathbf{y} = (\mathbf{q}, \mathbf{q}^\cdot)$, piecewise-continuous and 2π-periodic in t, characterizing an arbitrary Hamiltonian perturbation; $\mathbf{q}^\cdot \in \mathbf{R}^n$ is the vector of generalized velocities that determine the given non-Hamiltonian perturbation have the form

$$\mathbf{Q} = \varepsilon\mathbf{Gq}^\cdot + \varepsilon\mathbf{Fq} \tag{2.7}$$

For simplicity, the matrices $\mathbf{B}, \mathbf{C}, \mathbf{P}, \mathbf{G}, \mathbf{F}$ are assumed to be constant, $\det \mathbf{B} \neq 0$, $\det \mathbf{P} \neq 0$, $\mathbf{B}$ is positive-definite or sign-variable, and we may assume without loss of generality that $\mathbf{B}^\mathrm{T} = \mathbf{B}$, $\mathbf{C}^\mathrm{T} = -\mathbf{C}$, $\mathbf{P}^\mathrm{T} = \mathbf{P}$, $\mathbf{G}^\mathrm{T} = \mathbf{G}$, $\mathbf{F}^\mathrm{T} = -\mathbf{F}$. In accordance with the usual terminology [7], we will call the forces $\varepsilon\mathbf{Fq}$ non-conservative and the forces $\varepsilon\mathbf{Gq}^\cdot$ dissipative (definitely dissipative) if the matrix $(-\mathbf{G})$ is positive-semidefinite (positive-definite). Then $\mathbf{H}_{11} = \mathbf{B}^{-1}$, $\mathbf{H}_{12} = -\mathbf{B}^{-1}\mathbf{C}/2$, $\mathbf{H}_{22} = \mathbf{P} + \mathbf{C}^\mathrm{T}\mathbf{B}^{-1}\mathbf{C}/4$, $\mathbf{\Delta}_{11} = \mathbf{GB}^{-1}$, $\mathbf{\Delta}_{12} = \mathbf{F} - \mathbf{GB}^{-1}\mathbf{C}/2$, $\mathbf{\Delta}_{21} = \mathbf{\Delta}_{22} = \mathbf{O}$. Again putting $n = 2$ and assuming that $\beta_1 \neq \beta_2$, we find $c = 0, d = 0, e = 0, f = 0$ and also

$$\frac{a}{4\pi} = \beta_1(\mathbf{Gf}_1, \bar{\mathbf{f}}_1) - i(\mathbf{Ff}_1, \bar{\mathbf{f}}_1), \quad \frac{b}{4\pi} = \beta_2(\mathbf{Gf}_2, \bar{\mathbf{f}}_2) - i(\mathbf{Ff}_2, \bar{\mathbf{f}}_2), \quad \mathbf{f}_1 \neq 0, \quad \mathbf{f}_2 \neq 0 \tag{2.8}$$

where $\mathbf{f}_j$ are the projections of $\mathbf{e}_j$ on the complexification of the space $\mathbf{R}^n = \{\mathbf{q}\}, j = 1, 2$.

The next two theorems are proved using formulae (2.8) and Theorems 2 and 3 as well as Lemma 2 of [4].

Theorem 4. Consider Lagrange's equations of the second kind with Lagrangian (2.6) satisfying the above assumptions. Suppose that the generalized perturbing forces $\mathbf{Q} = \varepsilon\mathbf{Gq}^\cdot$ are definitely dissipative or possess the property $(\mathbf{Gf}_j, \bar{\mathbf{f}}_j) < 0, j = 1, 2$. Then, if the difference combination-resonance critical relations (2.1) hold, $N \neq 0$, the unperturbed system of equations is strongly stable under a given non-Hamiltonian perturbation if both matrices $\mathbf{B}$ and $\mathbf{P}$ are positive-definite, and strongly unstable otherwise. If the sum combination-resonance critical relations (2.2) hold, the system cannot be either strongly stable or strongly unstable with multipliers of the first or second classes.

Theorem 4 is an analogue of the third and fourth Thomson–Tait theorems [7]; it shows that these theorems also hold when there is an arbitrary sufficiently small Hamiltonian periodic parametric perturbation not only in the non-resonant case [4], but also in the case of difference combination resonance (2.1), $N \neq 0$.

Let us explain this in greater detail. If the matrices $\mathbf{B}$ and $\mathbf{P}$ are positive-definite, i.e. the unperturbed kinetic energy $(\mathbf{Bq}^\cdot, \mathbf{q}^\cdot)/2$ is positive-definite and the unperturbed potential energy $U = (\mathbf{Pq}, \mathbf{q})/2$ has a minimum at $\mathbf{q} = \mathbf{O}$, then the superposition of dissipative perturbing forces makes the system strongly stable under this non-Hamiltonian perturbation. But if either of the matrices $\mathbf{B}$ or $\mathbf{P}$ is not positive-definite, e.g. if the potential energy U does not have a minimum at $\mathbf{q} = \mathbf{O}$, and the unperturbed system is stabilized when there is no dissipation by gyroscopic forces, then the superposition of definitely dissipative perturbing forces makes the system strongly unstable.

Now suppose that the generalized forces (2.7) include non-conservative forces, i.e. $\mathbf{F} \neq \mathbf{O}$. Taking into account that when $\mathbf{C} = \mathbf{O}$ the vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ may be assumed to be real, we obtain the following proposition.

Theorem 5. Consider a system of Lagrange equations of the second kind with Lagrangian (2.6) satisfying all the assumptions formulated above and with generalized forces (2.7). Suppose that there are no gyroscopic forces of zero order, i.e. $\mathbf{C} = \mathbf{O}$ or $\mathbf{C} = \varepsilon\mathbf{C}_1$ and the resonance relations (2.1) with $N \neq 0$ or (2.2) hold. Then arbitrary given non-conservative perturbing forces $\varepsilon\mathbf{Fq}$ do not affect the strong stability or instability of the unperturbed system with multipliers of the first or second classes.

Corollary 1. If $\mathbf{C} = \mathbf{O}$ or $\mathbf{C} = \varepsilon\mathbf{C}_1$, Theorem 4 also holds for Lagrange equations of the second kind that include arbitrarily given non-conservative perturbing forces.

These results may be generalized to the case $n > 2$. The next theorem holds for arbitrary dimension $n \geqslant 1$. Suppose that the mechanical system admits of a gyroscopic interaction of zero order, i.e. $\mathbf{C} \neq \mathbf{O}$, and that it includes definitely dissipative and non-conservative perturbing generalized forces such that, on the unit sphere $|\mathbf{q}| = 1$

$$g = -\max(\mathbf{Gq}, \mathbf{q}) > 0, \quad h = \max(\mathbf{FG}^{-1}\mathbf{Fq}, \mathbf{q}) \geqslant 0 \tag{2.9}$$

Theorem 6. Consider Lagrange's equations of the second kind with Lagrangian (2.6), $(\mathbf{q}, \mathbf{q}^\cdot) \in \mathbf{R}^{2n}$ $(n \geqslant 1)$ and generalized forces (2.7), on the assumption that all the conditions analogous to those formulated above for $n = 2$ hold. Suppose that the perturbing generalized forces, which depend on the velocities, are definitely dissipative, i.e. the matrix $\mathbf{G}$ is negative-definite. Suppose moreover that $\beta_j + \beta_\nu \neq N, j, \nu = 1, 2, \ldots, n, N = 1, 2, \ldots$, i.e. there are no basic or combined-sum resonances of any multiplicity, $\beta_j \neq \beta_\nu$ for $j \neq \nu$, and moreover the least frequency $\beta = \min\{\beta_1, \ldots, \beta_n\} > 0$ satisfies the condition $h < \beta^2 g$. Then the equations are strongly stable for a given non-Hamiltonian perturbation if both matrices $\mathbf{B}$ and $\mathbf{P}$ are positive-definite and strongly unstable otherwise. If there are no non-conservative forces, $h = 0$, and the condition $h < \beta^2 g$ is always satisfied.

We add a few comments about Lagrange's equations of the type described in Theorem 6. Increasing the least natural frequency to a value $\beta > \sqrt{(h/g)}$, say, by increasing the stiffness [13] of the mechanical system, one can make the equations strongly stable or unstable under a given non-Hamiltonian perturbation, depending on whether both matrices $\mathbf{B}$ and $\mathbf{P}$ are positive-definite or not, provided that there are no critical relations of fundamental, simple or sum combination resonances.

In conclusion, we note that the concept of a non-Hamiltonian perturbation presupposes that the canonical variables $\mathbf{p}$ and $\mathbf{q}$ can be fixed. As we know, the Hamiltonian property of a system of differential equations is not invariant under a change of variables, even if fixed generalized coordinates are preserved. Thus, certain modifications of the Lagrangian may transform Lagrange's equations of the second kind into Lagrange's equations of the second kind with generalized forces of another type. Under such conditions the Lagrangian may lose its property of convexity with respect to $\mathbf{q}^\cdot$.

In view of the property of such preliminary transformations, we have omitted the requirement, quite common in mechanics, that the matrix $\mathbf{B}$ must be positive-definite. Finally, the equations of motion of certain linear holonomic mechanical systems with potential and definitely dissipative forces may also be reduced to Hamiltonian form (see, for example, [14]). However, under such transformations of definitely dissipative forces into potential forces, the corresponding functions H and L, which were originally independent of or periodically dependent on t, may lose that property, so that it becomes impossible to apply our strong stability and instability theorems.

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